Optimization methods without partial derivatives

Derivative Matching Method for Constrained Optimization

Suppose we want to optimize (maximize or minimize) an objective function $F(x, y)$ subject to a constraint $G(x, y) = C$. The idea is to assume that, at the optimal point, the value of the objective function is constant. This allows us to express one variable in terms of the other using both the objective function and the constraint. Then, we differentiate each expression with respect to the other variable, equate the derivatives, and from that equality, we obtain a relationship between x and y that, when substituted into the constraint, allows us to find the optimal solution.

The procedure can be summarized in the following steps:

1. **Fix the optimal level:** We assume that at the optimum, the objective function reaches a constant value, say $F(x, y) = K$.

2. **Solve for x (or y) as a function of y (or x):** We solve the equation $F(x, y) = K$ to obtain an expression of the form
$$x = f(y),$$
and we also solve for the same variable using the constraint:
$$x = g(y) \quad \text{(obtained from } G(x, y) = C\text{)}.$$

3. **Differentiate with respect to the other variable:** We differentiate both expressions with respect to y. This gives:
$$\left.\frac{dx}{dy}\right|_{\text{objective}} = f'(y) \quad \text{and} \quad \left.\frac{dx}{dy}\right|_{\text{constraint}} = g'(y).$$

4. **Match the derivatives:** At the optimal point, both curves are tangent, so their slopes must be equal. This implies:
$$f'(y) = g'(y).$$
This equality provides a relationship between x and y (or allows us to solve for y).

5. **Repeat the process with the other variable:** Similarly, we solve for y as a function of x using both the objective function and the constraint:
$$y = h(x) \quad \text{and} \quad y = k(x),$$
and we differentiate both expressions with respect to x:
$$\left.\frac{dy}{dx}\right|_{\text{objective}} = h'(x) \quad \text{and} \quad \left.\frac{dy}{dx}\right|_{\text{constraint}} = k'(x).$$
Equating these derivatives yields another relationship between x and y. This second equality, together with the previous one, allows us to eliminate the parameter K (the fixed value of the objective function) and obtain a unique relationship between x and y.

6. **Substitute into the constraint:** Using the relationship obtained, we substitute into the original constraint $G(x, y) = C$ to compute the optimal value of one of the variables, and from there, the other.

Example: Optimization of $f(x,y) = x^2 y$ subject to $5x + 2y = 100$

Let the objective function be

$$f(x,y) = x^2 y,$$

and the constraint

$$5x + 2y = 100.$$

At the optimum, we assume that $f(x,y) = K$ is constant.

Solving for x as a function of y

- **From the objective function:** Starting with

$$x^2 y = K,$$

  we solve for x (taking the positive branch):

$$x = \sqrt{\frac{K}{y}} = \sqrt{K}\, y^{-1/2}.$$

- **From the constraint:** From

$$5x + 2y = 100,$$

  we solve for x:

$$x = \frac{100 - 2y}{5}.$$

- **Differentiate with respect to y:**
  - For the objective function:

$$\frac{dx}{dy} = -\frac{1}{2}\sqrt{K}\, y^{-3/2}.$$

  - For the constraint:

$$\frac{dx}{dy} = -\frac{2}{5}.$$

- **Match the derivatives:** At the optimum, we have:

$$-\frac{1}{2}\sqrt{K}\, y^{-3/2} = -\frac{2}{5}.$$

  Canceling the negative sign and solving for $\sqrt{K}$:

$$\frac{1}{2}\sqrt{K}\, y^{-3/2} = \frac{2}{5} \quad \Longrightarrow \quad \sqrt{K} = \frac{4}{5}\, y^{3/2}.$$

  Squaring both sides:

$$K = \frac{16}{25}\, y^3.$$

Solving for y as a function of x

- **From the objective function:** Starting from

$$x^2 y = K,$$

  we solve for y:

$$y = \frac{K}{x^2}.$$

- **From the constraint:** From
$$5x + 2y = 100,$$
we solve for y:
$$y = \frac{100 - 5x}{2}.$$
- **Differentiate with respect to x:**
  - For the objective function:
$$y = K\,x^{-2} \quad\Longrightarrow\quad \frac{dy}{dx} = -2K\,x^{-3}.$$
  - For the constraint:
$$\frac{dy}{dx} = -\frac{5}{2}.$$
- **Match the derivatives:** Equating:
$$-2K\,x^{-3} = -\frac{5}{2}.$$
Canceling the negative sign and solving for K:
$$2K\,x^{-3} = \frac{5}{2} \quad\Longrightarrow\quad K = \frac{5}{4}\,x^3.$$

Equating values of K and obtaining the relationship between x and y

From both processes we obtained:
$$K = \frac{16}{25}\,y^3 \quad \text{and} \quad K = \frac{5}{4}\,x^3.$$
Equating:
$$\frac{16}{25}\,y^3 = \frac{5}{4}\,x^3.$$
Multiplying both sides by 100 (or simplifying directly):
$$\frac{64}{100}\,y^3 = \frac{125}{100}\,x^3,$$
which simplifies to:
$$64\,y^3 = 125\,x^3.$$
Dividing both sides by x^3:
$$\left(\frac{y}{x}\right)^3 = \frac{125}{64},$$
and taking the cube root:
$$\frac{y}{x} = \frac{5}{4} \quad\Longrightarrow\quad y = \frac{5}{4}\,x.$$

Substitution into the constraint to find x and y

Substituting $y = \frac{5}{4}\,x$ into the constraint
$$5x + 2y = 100,$$
we get:
$$5x + 2\left(\frac{5}{4}\,x\right) = 5x + \frac{10}{4}\,x = 5x + \frac{5}{2}\,x = \frac{15x}{2} = 100.$$
Solving for x:
$$x = \frac{100 \cdot 2}{15} = \frac{200}{15} = \frac{40}{3}.$$
Then, y is:
$$y = \frac{5}{4}\,x = \frac{5}{4} \cdot \frac{40}{3} = \frac{200}{12} = \frac{50}{3}.$$

Optimal value of the function

Finally, the optimal value of the function is:

$$f\left(\frac{40}{3}, \frac{50}{3}\right) = \left(\frac{40}{3}\right)^2 \cdot \frac{50}{3} = \frac{1600}{9} \cdot \frac{50}{3} = \frac{80000}{27}.$$

Method summary

1. Solve for x in terms of y (and vice versa) from the objective function $x^2y = K$ and the constraint $5x + 2y = 100$.
2. Differentiate both expressions with respect to the corresponding variable and equate them to obtain two expressions involving K.
3. Equate the values of K obtained in both processes to eliminate the parameter K and find a unique relationship between x and y.
4. Finally, substitute this relationship into the constraint to determine the optimal values of x and y.

Example: Minimization of $C = 3x_1 + 6x_2 + 180$ subject to $x_1x_2 = 2$

We want to minimize

$$C = 3x_1 + 6x_2 + 180,$$

subject to the constraint

$$x_1x_2 = 2.$$

We fix a constant level for the objective function:

$$3x_1 + 6x_2 = K,$$

Step 1: Differentiating the Objective Function

We solve for x_2 in terms of x_1:

$$x_2 = \frac{K - 180 - 3x_1}{6}.$$

Differentiating with respect to x_1, we obtain:

$$\frac{dx_2}{dx_1} = -\frac{3}{6} = -\frac{1}{2}.$$

Note that the derivative does not depend on K (the constant disappears).

Step 2: Differentiating the Constraint

From the constraint

$$x_1x_2 = 2,$$

we solve for x_2:

$$x_2 = \frac{2}{x_1}.$$

Differentiating with respect to x_1, we get:

$$\frac{dx_2}{dx_1} = -\frac{2}{x_1^2}.$$

Step 3: Matching the Derivatives

At the point of tangency between the level curve of the objective function and the constraint curve, their slopes must be equal. That is:

$$-\frac{1}{2} = -\frac{2}{x_1^2}.$$

Canceling the negative sign and solving:

$$\frac{1}{2} = \frac{2}{x_1^2} \quad \Longrightarrow \quad x_1^2 = 4 \quad \Longrightarrow \quad x_1 = 2,$$

(assuming $x_1 > 0$).

Step 4: Determine x_2 and the Optimal Cost

Substituting $x_1 = 2$ into the constraint:

$$x_2 = \frac{2}{2} = 1.$$

Therefore, the minimum cost is:

$$C_{\min} = 3(2) + 6(1) + 180 = 6 + 6 + 180 = 192.$$

Substitution Method for Constrained Optimization

Suppose we want to optimize (maximize or minimize) an objective function $F(x, y)$ subject to a constraint $G(x, y) = C$. This method consists of solving the constraint for one variable, substituting that expression into the objective function to reduce the problem to a single variable, differentiating with respect to that variable, setting the derivative equal to zero, and finally solving to find the optimal solution.

The procedure can be summarized in the following steps:

1. **Solve the constraint:** Given the constraint

$$G(x, y) = C,$$

we solve for one of the variables. For example, if we solve for y, we get:

$$y = g(x).$$

2. **Substitute into the objective function:** We substitute $y = g(x)$ into the objective function $F(x, y)$ to obtain a function of a single variable:

$$f(x) = F(x, g(x)).$$

3. **Differentiate and set equal to zero:** We differentiate the function $f(x)$ with respect to x and impose the first-order condition:

$$\frac{df(x)}{dx} = 0.$$

We solve this equation to find the optimal value x^*.

4. **Determine the other variable:** Finally, we substitute x^* into the expression $y = g(x)$ to obtain y^*.

This method reduces the original two-variable problem to a single-variable problem, simplifying the optimization process.

Example: Optimization of $f(x,y) = x^2y$ subject to $5x+2y=100$ (Substitution Method)

Let the objective function be

$$f(x,y) = x^2y,$$

and the constraint

$$5x+2y=100.$$

The substitution method consists of solving for one variable from the constraint, inserting that expression into the objective function to obtain a function of a single variable, differentiating with respect to that variable, setting the derivative equal to zero, and finally solving.

Step 1: Solve the constraint

From the equation

$$5x+2y=100,$$

we solve for y:

$$y = \frac{100-5x}{2}.$$

Step 2: Substitute into the objective function

We insert the expression for y into $f(x,y) = x^2y$:

$$f(x) = x^2\left(\frac{100-5x}{2}\right) = \frac{1}{2}\left(100x^2 - 5x^3\right).$$

Step 3: Differentiate and set to zero

We differentiate $f(x)$ with respect to x:

$$f'(x) = \frac{1}{2}(200x - 15x^2) = 100x - \frac{15}{2}x^2.$$

To find the optimum, we set the derivative equal to zero:

$$100x - \frac{15}{2}x^2 = 0.$$

Factoring out x:

$$x\left(100 - \frac{15}{2}x\right) = 0.$$

We obtain two solutions: $x=0$ (trivial solution) and

$$100 - \frac{15}{2}x = 0 \quad \Longrightarrow \quad x = \frac{200}{15} = \frac{40}{3}.$$

We discard $x=0$ and take $x^* = \frac{40}{3}$.

Step 4: Determine y

Substituting $x^* = \frac{40}{3}$ into the expression for y:

$$y^* = \frac{100 - 5\left(\frac{40}{3}\right)}{2} = \frac{100 - \frac{200}{3}}{2} = \frac{\frac{100}{3}}{2} = \frac{50}{3}.$$

Step 5: Optimal value of the function

The optimal value of the function is:

$$f\left(\frac{40}{3}, \frac{50}{3}\right) = \left(\frac{40}{3}\right)^2 \cdot \frac{50}{3} = \frac{80000}{27}.$$

Summary of the Substitution Method

1. Solve one variable from the constraint, in this case $y = \frac{100-5x}{2}$.
2. Substitute into the objective function to obtain $f(x) = \frac{1}{2}(100x^2 - 5x^3)$.
3. Differentiate $f(x)$ with respect to x and set the derivative equal to zero to find the optimal value x^*.
4. Use x^* in the expression for y to determine y^*.
5. Evaluate the objective function at (x^*, y^*) to obtain the optimal value.

Example: Minimization of $C = 3x_1 + 6x_2 + 180$ subject to $x_1x_2 = 2$ (Substitution Method)

We want to minimize

$$C = 3x_1 + 6x_2 + 180,$$

subject to the constraint

$$x_1x_2 = 2.$$

Step 1: Solve for one variable from the constraint

From the constraint $x_1x_2 = 2$, we solve for x_2:

$$x_2 = \frac{2}{x_1}.$$

Step 2: Substitute into the objective function

We substitute the expression for x_2 into the simplified objective function:

$$\widetilde{C}(x_1) = 3x_1 + 6\left(\frac{2}{x_1}\right) = 3x_1 + \frac{12}{x_1} + 180.$$

Step 3: Differentiate and set to zero

We differentiate $\widetilde{C}(x_1)$ with respect to x_1:

$$\frac{d\widetilde{C}}{dx_1} = 3 - \frac{12}{x_1^2}.$$

To find the minimum, we set the derivative equal to zero:

$$3 - \frac{12}{x_1^2} = 0.$$

Solving:

$$\frac{12}{x_1^2} = 3 \quad \Longrightarrow \quad x_1^2 = 4,$$

so, assuming $x_1 > 0$,

$$x_1^* = 2.$$

Step 4: Determine x_2 and the optimal cost

Substituting $x_1 = 2$ into the expression for x_2:

$$x_2^* = \frac{2}{2} = 1.$$

The minimum cost is therefore:

$$C_{\min} = 3(2) + 6(1) + 180 = 6 + 6 + 180 = 192.$$

Summary of the Substitution Method

1. Solve for x_2 from the constraint $x_1 x_2 = 2$: $x_2 = \frac{2}{x_1}$.
2. Substitute into the simplified objective function $\widetilde{C} = 3x_1 + 6x_2$ to obtain
$$\widetilde{C}(x_1) = 3x_1 + \frac{12}{x_1}.$$
3. Differentiate with respect to x_1 and set the derivative equal to zero to find x_1^*.
4. Use the constraint expression to determine x_2^* and evaluate C.

Corner Solutions

It is important to note that the substitution method (or derivative matching method) assumes that the optimal solution is interior—that is, the level curve of the objective function is tangent to the constraint. This method is not applicable when the optimum occurs at a corner solution.

Brief example: Consider the maximization of

$$f(x, y) = x + y,$$

subject to the constraints

$$\begin{cases} x + y \leq 1, \\ x \geq 0, \quad y \geq 0. \end{cases}$$

In this problem, the optimum is reached at the corners $(0, 1)$ or $(1, 0)$, where there is no tangency between the level curve and the boundary. Therefore, methods based on derivative matching are not suitable in this case.